

Appendix: “Volatility Factor in Concept and Practice”

April 8, 2011

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Note: The material contained here is supplementary to the article named in the title. The article is published in May 2011 Issue of Today Magazine, the official publication of National Crop Insurance Services (NCIS), Overland Park, KS. The article and supplementary material is available at the website: <http://www.ag-risk.org/NCISPub.htm>.

Appendix 1: A Technical Treatment of Volatility

Technically, volatility is defined as the standard deviation of a certain type of return to a futures price over a year. Denote the current level of futures price with P_0 and its level at a future time period (T) with P_T . Here current is represented by subscript zero and refers to a point in time where one can see a year out before futures contract expires. Over a year $T = 1$ and P_T becomes P_1 . Note that, P_1 is a random variable, that is, its value is uncertain at time $T = 0$. The particular type of return that will be used here is called the ‘logarithmic return’ and is obtained as defined as the natural logarithm of the ratio of P_1 to P_0 . Volatility is then defined as

$$\sigma \equiv SDEV(\ln(P_1/P_0)) \quad (1)$$

where \equiv means defined by, $SDEV$ denotes the standard deviation, and $\ln(\cdot)$ is the natural logarithm function. To economize on the notation, we will use $Z_1 \equiv \ln(P_1/P_0)$. Z_1 is interpreted as the return (at a continuously compounded rate; denote with z) realized over a year. For small changes between P_1 and P_0 , Z_1 is approximately equal the percentage change in futures price over a year.

So far what is referred and described as “volatility” is the annualized volatility. The volatility over different time periods can be similarly defined. Consider a future time period T from a given point in time $T = 0$ where T can be shorter, equal or longer than a year. Define the natural logarithm of the ratio of P_T to P_0 with Z_T , that is,

$$Z_T \equiv \ln(P_T/P_0) \tag{2}$$

Note that Z_1 defined earlier is a particular case of Z_T when $T = 1$. Similarly, Z_T is the return (at a continuously compounded rate z) realized over a time period T and expressed as:

$$P_T = e^{zT} P_0 \tag{3}$$

where e is exponential number, 2.71828, raised to the power of z times T .¹ Applying the natural logarithm to the both sides of equation would yield

$$z \times T = \ln(P_T/P_0). \tag{4}$$

Comparing equations (2) and (4), Z_T is simply a short-hand notation for $z \times T$. Note that z is the rate of growth at a point in time (instantaneous) while $Z_T = z \times T$ is the rate of growth over a period of time between $T = 0$ and T .

Under certain analytical assumptions about the behavior of futures price through time (whose discussion is beyond our scope here; see Hull, Chapter 12), Z_T can be derived as

¹ Economic interpretation of this number is the following: \$1 will grow into a value of $\$e = \2.71828 at the continuously compounded nominal interest rate of 100% per annum (see page 277 in Chiang.)

normally distributed with the mean $(\mu - \sigma^2/2)T$ and standard deviation of $\sigma\sqrt{T}$. Note that in the preceding statement, μ denotes the expected rate of return for the futures price per year and σ as defined earlier (the volatility per year).² Therefore, the volatility over a time period between $T = 0$ and T is $\sigma\sqrt{T}$, the standard deviation of Z_T .

Therefore, the volatility for any period of time is $\sigma\sqrt{T}$, where \sqrt{T} is the time adjustment factor. Suppose $\sigma = 30\%$. The longer (shorter) the time period, the higher (lower) the volatility is expected to be. The time-adjustment factor (\sqrt{T}) becomes $\sqrt{2} = 1.41$, $\sqrt{1} = 1$, $\sqrt{0.5} = 0.71$, $\sqrt{1/52} = 0.14$, and $\sqrt{1/365} = 0.0027$ for two-years, one-year, six-months, one week, and one day, respectively. Note that the resulting standard deviation over a day ($\sigma\sqrt{1/365} = \sigma \times 0.0027$) has to be quite lower relative to σ (the standard deviation over a year; i.e. volatility).

The following example demonstrates a use of knowing the distribution of Z_T when $T = 1$. Note that all the notation in this example is as defined earlier, yet some of these definitions will be repeated. Assume that $\sigma = 0.3$ or 30%, $P_0 = \$6$, $\mu = 14.5\%$ (chosen for the sake of convenience). Then, Z_1 is normally distributed with the mean $\mu - \sigma^2/2$ which takes the value of $0.145 - 0.3^2/2 = 0.1$, that is, 10%, and the standard deviation $\sigma\sqrt{1} = \sigma = 0.3$. That implies Z_1 will be between -48.8% and 68.5% with 95% probability. Note that the lower bound (-48.8%) is

² Note that the distribution of Z_T is actually based on the distribution of instantaneous rate of growth z because $Z_T = z \times T$ and T is the time-period and known in advance. By using the relationship $Z_T = z \times T$ and the stated distribution for Z_T , one can verify that z is normally distributed with the mean $(\mu - \sigma^2/2)$ and the standard deviation σ/\sqrt{T} .

1.96 times the standard deviation ($1.96 \times 30\% = 58.8\%$) below the mean (10%), whereas the upper bound (68.8%) is 1.96 times the standard deviation above 10%.

The preceding example can be extended to arrive at a confidence interval for the level of futures price over a year (P_1). That requires obtaining the distribution of $\ln(P_1)$ based on the distribution of Z_1 . To this end, we will first give the distribution of $\ln(P_T)$ for a general time period T and the associated Z_T . One can re-arrange equation (2) and obtain:

$$\ln(P_T) = Z_T - \ln(P_0) \quad (5)$$

Recall that Z_T is normally distributed with the mean $(\mu - \sigma^2/2)T$ and standard deviation of $\sigma\sqrt{T}$. Combining that with the relationship in equation (5), one obtains the distribution of $\ln(P_T)$ as normally-distributed with the mean $\ln(P_0) + (\mu - \sigma^2/2)T$ and the standard deviation $\sigma\sqrt{T}$. Note that the standard deviations of $\ln(P_T)$ and Z_T are the same from equation (5).³

Using the distribution of $\ln(P_T)$, $\ln(P_1)$ is then distributed with the mean $\ln(P_0) + (\mu - \sigma^2/2)$ and standard deviation σ . Note that $\ln(P_0) = 1.792$ as $P_0 = \$6$. Then, the 95% confidence interval for $\ln(P_1)$ can be easily obtained from the previously found 95% interval for Z_1 by adding $\ln(P_0) = 1.792$ to the lower and upper bound of the latter interval as: $1.304 = 1.792 - 0.488$ and $2.479 = 1.792 + 0.688$. Finally, by taking the exponential power of the lower and upper bounds, one arrives at the confidence interval for P_1 as $3.683 = e^{1.304}$ and

³ The given distribution of $\ln(P_T)$ is known as the ‘log-normal’ property of futures prices, that is, the distribution of P_T becomes normal once it is transformed by natural logarithm $\ln(\cdot)$ function. The literature seems to reach consensus over the choice of log-normal distribution in simulating the futures prices.

$11.94 = e^{2.479}$ as the lower and upper bound, respectively. Thus, for a futures price (currently at \$6 per bushel with the expected return of $\mu = 14.5\%$ per annum and standard deviation of 30% per annum), there is a 95% probability that the futures price will lie between \$3.683 and \$11.94 per bushel over a year.

Appendix 2: Derivation of Black-Sholes Formulas

This section lays out the key steps involved in the derivations of Black-Sholes formulas. In addition to the notation introduced so far, note the following: K is the strike price; r is the continuously compounded risk-free rate of interest; T is the time to maturity of the option, $N(x)$ is the probability that a normal random variable with mean 0 and standard deviation 1 will be less than x .⁴

Consider a call option on a futures price maturing at time T . A call option gives the right (but not the obligation) to buy the crop at the strike price. The call option will have an intrinsic value so long as the future value of the futures price (P_T) remains higher than the strike price (K). The expected (statistical) value of call option can be expressed as $E[\max(P_T - K, 0)]$ where $E[.]$ is the expectation operator. Discounting the expected value into current dollars at the rate of r yield the price of the call option as

$$c \equiv e^{-rT} \times E[\max(P_T - K, 0)] \quad (6)$$

The distribution of P_T is given as log-normal as before. Taking the expectation over the probability density function of P_T yields the expected value of a call option (see the Appendix in Hull, page 307) as

$$E[\max(P_T - K, 0)] = E(P_T) \times N(x_1) - K \times N(x_2) \quad (7)$$

⁴ Such a normal random variable is called the standard normal random variable. For a particular value of x , $N(x)$ can be calculated from the Excel command =NORMSDIST(). For example, for $x=0$, the command =NORMSDIST(0) will produce the value of 0.5, which is expected because the standard normal distribution is symmetric around zero.

where $E(P_T)$ is the expected value of futures price, x_1 and x_2 are specific threshold values and $N(x_1)$ and $N(x_2)$ are the corresponding probabilities obtained from the standard normal distribution (see footnote 4).

Note Black-Sholes formulas are based on risk-neutral valuation (that is, all investors are assumed to be risk-neutral; that is, the expected payoff maximizers).⁵ In such an environment, the expected return to a futures price (denoted with μ earlier) would be equal to the return from risk-free asset (r).⁶ Then, the expected value of P_T can be obtained as

$$E(P_T) = P_0 e^{rT} \quad (8)$$

where the initial price P_0 is continuously compounded at the risk-free rate of return (r) over a time period T .

Plugging the expression for $E(P_T)$ from equation (8) into equation (7) and threshold values x_1 and x_2 ; and further plugging the resulting expressions in equation (6) yields Black-Sholes final formula for pricing the call option:

$$c = P_0 N(x_1) - Ke^{-rT} N(x_2) \quad (9)$$

where

⁵ Risk-neutrality assumption buys a great deal of simplification in the derivations. Despite the fact that Black-Sholes formulas are obtained in a risk-neutral world, they are applicable to other risk environments. Moving from a risk-neutral to risk-averse world changes the expected growth rate and discounting factor at the same time and both effects cancel each other (Hull, page 290).

⁶ By the time this article was written, the risk-free interest rate per annum was around 0.15% in calculation of implied volatilities by barchart.com.

$$x_1 = \frac{\ln(P_0) + rT - \ln(K) + \sigma^2 T/2}{\sigma\sqrt{T}} \quad (10)$$

$$x_2 = \frac{\ln(P_0) + rT - \ln(K) - \sigma^2 T/2}{\sigma\sqrt{T}} = x_1 - \sigma\sqrt{T} \quad (11)$$

Consider a put option on a futures price maturing at time T . Similarly, a put option gives the right (but not the obligation) to sell the crop at the strike price. A put option will have an intrinsic value so long as the value of the futures price in the future remains lower than the strike price of the put option. The expected (statistical) value of call option can be expressed as $E[\max(K - P_T, 0)]$. Discounting the expected value into current dollars at the rate of r yield the price of the put option (denote with d) as

$$d \equiv e^{-rT} \times E[\max(K - P_T, 0)] \quad (12)$$

Similar to equation (7), the expected value of an put option can be obtained as

$$E[\max(K - P_T, 0)] = K \times N(-x_2) - E(P_T) \times N(-x_1) \quad (13)$$

where x_1 and x_2 are given in equations (10) and (11). Plugging $E(P_T)$ from equation (8) into equation (13), and further plugging the resulting equation in equation (12), gives the pricing equation for a put option as

$$d \equiv e^{-rT} [K \times N(-x_2) - P_0 e^{rT} \times N(-x_1)] \quad (14)$$

where x_1 and x_2 are given in equations (10) and (11).

Regarding the relative magnitudes of prices of call and put options, the following relationship can be obtained. Because the standard normal distribution is symmetric (see footnote 4), it follows that $N(-x_2) = 1 - N(x_2)$ and $N(-x_1) = 1 - N(x_1)$. By plugging the preceding relations in equation (14) and re-arranging the terms in line with equation (13), one can obtain the following relationship between the call and put options prices:

$$c = [P_0 - e^{-rT} K] + d \quad (15)$$

The preceding relation means the following: if the current value of the futures price is high enough to exceed the present value (the value discounted at the risk-free interest rate) of the strike price of the option (that is, $P_0 > e^{-rT} K$), then the price of call option should be higher than the price of put option ($c > d$) from equation (15). Intuitively, if the current value of the futures price increases relative to the strike price, it is more likely that the call option will be exercised. If, on the other hand, $P_0 < e^{-rT} K$, then $c < d$. Finally, $P_0 = e^{-rT} K$ would imply $c = d$.

References:

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